◇ Monograph Excerpt ◇

# Matter Antimatter Fluctuations

## search, discovery and analysis of Bs flavor oscillations

Nuno Leonardo

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## 2.1 Quantum mechanics of particle-antiparticle oscillations

The quantum mechanical formalism of particle-antiparticle oscillations is here exposed. While the focus will in time be placed on the $B\bar{B}$ system, we start from a more general situation where particle and antiparticle are distinguished by an internal quantum number, $F$, like beauty, strangeness, charm, lepton number, baryon number, *etc.* The charged mesons, for instance $B^+/B^-$ and $B_c^+/B_c^-$, are not considered, as electric charge violation is not contemplated; unlike those other symmetries, electric charge conservation is protected by gauge, thus exact, symmetry. The $\pi^0$ mesons constitute their own antiparticles and are thus also excluded.

The formalism applies to the neutral $B$ meson systems, $B_s\bar{B}_s$ and $B^0\bar{B}^0$, studied in this monograph, and holds similarly for the $K^0\bar{K}^0$, $D^0\bar{D}^0$, neutron/antineutron and neutrino/antineutrino systems. Such a generic system will be here denoted by $P^0\bar{P}^0$, where $P^0$ and $\bar{P}^0$ are flavor eigenstates, distinguished merely by the internal quantum number $F$.

### 2.1.1 Effective Hamiltonian

An unstable particle can be described by a Hamiltonian, $\mathcal{H} = m - \frac{i}{2}\Gamma$, through the non-relativistic Schrödinger equation $i\partial_t\psi = \mathcal{H}\psi$. The solution

$$|\psi\rangle_t = e^{-imt}e^{-\frac{1}{2}\Gamma t}|\psi_0\rangle \tag{2.1}$$

reproduces the exponential law of radioactive decay, as $|\langle\psi_0|\psi\rangle_t|^2 = e^{-\Gamma t}$, with lifetime $\tau \equiv 1/\Gamma$. The Hamiltonian is not real (*i.e.* hermitian), since it describes the decay of a particle by its vanishing.

The $P^0\bar{P}^0$ pair can be described similarly as a decaying two-component quantum state. The effective Hamiltonian of the system will be formed of a component $\mathcal{H}_0$ which preserves the characteristic quantum number ($\Delta F = 0$) along with a component inducing $\Delta F \neq 0$ transitions; this can be written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\Delta F} \ . \tag{2.2}$$

Particle-antiparticle transitions are $\Delta F = 2$ processes which are induced by $\mathcal{H}_{\Delta F}$.

An arbitrary state of the system is represented by a vector in the Hilbert space as

$$|\psi\rangle = a\,|P^0\rangle + b\,|\bar{P}^0\rangle \ . \tag{2.3}$$

Its dynamics, in the Weisskopf-Wigner approximation [93], is determined by the time-dependent Schrödinger equation

$$i\,\frac{d}{dt}\psi = \mathcal{H}\,\psi\,. \tag{2.4}$$

The Hamiltonian contains a dispersive part and an absorptive part, its matrix representation being decomposed accordingly as $\mathbf{H} = \mathbf{M} - \frac{i}{2}\mathbf{\Gamma}$, with $\mathbf{M}$ and $\mathbf{\Gamma}$ complex hermitian matrices. Working in the flavor basis, (2.4) may be expressed as

$$i\,\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} m - \frac{i}{2}\Gamma & M_{12} - \frac{i}{2}\Gamma_{12} \\ M_{12}^* - \frac{i}{2}\Gamma_{12}^* & m - \frac{i}{2}\Gamma \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.5}$$

The diagonal Hamiltonian matrix elements describe the mass and decay width of the flavor eigenstates. CPT invariance, which is a basic feature of any local quantum field theory, guarantees equality of mass and lifetime of particles and antiparticles, leading to $M_{11} = M_{22} = m$ and $\Gamma_{11} = \Gamma_{22} = \Gamma$. The off-diagonal elements are responsible for $P^0\bar{P}^0$ transitions, where $M_{12}$ represents virtual transitions and $\Gamma_{12}$ represents real transitions through common decay modes.

The non-zero off-diagonal elements of the Hamiltonian matrix imply that the flavor eigenstates differ from the mass eigenstates. The latter will be referred to as *heavy (H)* and *light (L)* mass eigenstates, which are defined as

$$\begin{aligned} |P_L\rangle &= p\,|P^0\rangle + q\,|\bar{P}^0\rangle\,, \\ |P_H\rangle &= p\,|P^0\rangle - q\,|\bar{P}^0\rangle\,, \end{aligned} \tag{2.6}$$

with the complex coefficients $p$ and $q$ obeying the normalization condition $|p|^2 + |q|^2 = 1$. It should be noted that the states in (2.6) do not in general form an orthogonal set, as $\langle P_H|P_L\rangle = |p|^2 - |q|^2$ does not vanish if $|\frac{q}{p}| \neq 1$. In fact, this latter condition would mean that $P^0 \to \bar{P}^0$ and $\bar{P}^0 \to P^0$ transitions would occur at different rates (as it will become apparent below, see (2.18)), corresponding to CP violation in the mixing process. If CP is conserved, which occurs if $|\frac{q}{p}| = 1$ and $\arg\left(\frac{q}{p}\right) = 0$, in which case $q = p = \frac{1}{\sqrt{2}}$ up to some arbitrary phase convention, the mass and CP eigenstates coincide,

$$|P_{L,H}\rangle = \frac{|P^0\rangle \pm |\bar{P}^0\rangle}{\sqrt{2}} \qquad \text{with} \qquad \mathrm{CP}|P_{L,H}\rangle = \pm|P_{L,H}\rangle\,, \tag{2.7}$$

using a convention where $\mathrm{CP}\,|P^0\rangle = |\bar{P}^0\rangle$.

Solving the eigenvalue problem, $\det(\mathcal{H} - \lambda) = 0$, we obtain

$$\begin{aligned} \lambda_{H,L} &= m - \frac{i}{2}\Gamma \pm Q\,, \quad \text{with} \\ Q &\equiv \sqrt{\left(M_{12}^* - \frac{i}{2}\Gamma_{12}^*\right)\left(M_{12} - \frac{i}{2}\Gamma_{12}\right)}\,. \end{aligned} \tag{2.8}$$

The eigenvalues can be further expressed as

$$\lambda_{H,L} = m_{H,L} - \frac{i}{2}\Gamma_{H,L}\,, \quad \text{with} \tag{2.9}$$

$$m_{H,L} = \mathrm{Re}\,(\lambda_{H,L})\,, \qquad \Gamma_{H,L} = -\,2\,\mathrm{Im}\,(\lambda_{H,L})\,. \tag{2.10}$$

It is immediately seen that $m$ and $\Gamma$ are the average mass $\frac{1}{2}\,(m_H + m_L)$ and width $\frac{1}{2}\,(\Gamma_H + \Gamma_L)$. In order to obtain the explicit relationship between the matrix elements, $M_{12}$ and $\Gamma_{12}$, and the observables

$$\Delta m \equiv m_H - m_L\,, \tag{2.11}$$

$$\Delta\Gamma \equiv \Gamma_L - \Gamma_H\,, \tag{2.12}$$

we can form the quantity $(\lambda_H - \lambda_L)^2 = (2Q)^2$ to find, from (2.8) above,

$$\begin{aligned}
(\Delta m)^2 - \frac{1}{4}\,(\Delta\Gamma)^2 &= 4|M_{12}|^2 - |\Gamma_{12}|^2\,, \\
\Delta m\,\Delta\Gamma &= -4\,\mathrm{Re}\,(M_{12}\Gamma_{12}^*)\,, \\
\frac{q}{p} &= -\frac{\Delta m + i\,\Delta\Gamma/2}{2M_{12} - i\,\Gamma_{12}} = -\frac{2M_{12}^* - i\,\Gamma_{12}^*}{\Delta m + i\,\Delta\Gamma/2} = \sqrt{\frac{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}{M_{12} - \frac{i}{2}\Gamma_{12}}}\,.
\end{aligned} \tag{2.13}$$

**Mechanical analogon**

The mechanical system formed of two coupled, identical pendula is also characterized by (2.5) [94]. In the absence of the coupling, they would be both described by an oscillation frequency $m$ and a damping constant $\Gamma$. The two pendula correspond to the particle $P^0$ and antiparticle $\bar{P}^0$, in this case governed by $\mathcal{H}_0$ before the perturbation.

Once they are coupled by a spring characterized by elasticity proportional to $M_{12}$ and damping constant $\Gamma_{12}$ the solutions will comprise two eigenstates: *(i)* corresponding to a long-lived (*i.e.* low damping), light (*i.e.* low frequency) state, where the pendula oscillate in phase; and *(ii)* corresponding to a short-lived (*i.e.* high damping), heavy (*i.e.* high frequency) mode with a phase difference of 180°. The differences in frequency and damping for the two modes are given by $\Delta m = 2M_{12}$ and $\Delta\Gamma = 2\Gamma_{12}$.

As one pendulum is excited, it will transfer its energy to the other, and back, producing a beat with frequency $2\pi f_{12} = \Delta m$. This beat corresponds analogously to the oscillation between a particle $P^0$ and its antiparticle $\bar{P}^0$, where the mass difference $\Delta m$ is actually observed as a frequency.

It should be noted, however, that in the mechanical system, unlike the case of the oscillating particles, $M_{12}$ and $\Gamma_{12}$ are strictly non-negative real numbers, and that the absence of non-trivial phases further prevents the system from simulating CP violation.

### 2.1.2 Time evolution

The solution to the system of coupled differential equations (2.5) is decomposed in single particle solutions (2.1) for the mass eigenstates. The latter evolve in time according to the corresponding eigenvalues found in (2.8),

$$|P_{L,H}\rangle_t \;=\; e^{-i\,\lambda_{L,H}\,t}\,|P_{L,H}\rangle \;=\; e^{-i\,m_{L,H}\,t-\frac{1}{2}\Gamma_{L,H}\,t}\,|P_{L,H}\rangle\;. \tag{2.14}$$

An arbitrary initial state of the system may be expressed as a linear combination of either flavor or mass eigenstates,

$$\begin{aligned} |\psi\rangle \;&=\; a_0\,|P^0\rangle + b_0\,|\bar{P}^0\rangle \;=\; \alpha_L\,|P_L\rangle + \alpha_H\,|P_H\rangle\,, \\ &\text{with}\qquad \alpha_{L,H} = \frac{1}{2}\left(\frac{a_0}{p} \pm \frac{b_0}{q}\right) \qquad \text{and} \\ &a_0 \;=\; p\,(\alpha_L+\alpha_H)\,, \qquad b_0 \;=\; q\,(\alpha_L-\alpha_H)\;. \end{aligned} \tag{2.15}$$

Its time evolution follows from (2.14), being given by

$$|\psi\rangle_t \;\equiv\; \mathcal{H}\,|\psi\rangle \;=\; \alpha_L\,|P_L\rangle_t + \alpha_H\,|P_H\rangle_t\;. \tag{2.16}$$

In particular, a state which is initially a pure flavor eigenstate (either $a_0 = 0$ or $b_0 = 0$) will evolve to a state of flavor admixture. Specifically, the time evolution of pure flavor eigenstates, as may be derived from the expressions above, is given by

$$\begin{aligned} |P^0\rangle_t \;&=\; \frac{1}{2p}\,(|P_L\rangle_t + |P_H\rangle_t) \;=\; g_+(t)\,|P^0\rangle + \frac{q}{p}\,g_-(t)\,|\bar{P}^0\rangle\,, \\ |\bar{P}^0\rangle_t \;&=\; \frac{1}{2q}\,(|P_L\rangle_t - |P_H\rangle_t) \;=\; \frac{p}{q}\,g_-(t)\,|P^0\rangle + g_+(t)\,|\bar{P}^0\rangle\,, \end{aligned} \tag{2.17}$$

where

$$g_\pm(t) \;=\; \frac{1}{2}\left[e^{-\left(im_L+\frac{1}{2}\Gamma_L\right)t} \pm e^{-\left(im_H+\frac{1}{2}\Gamma_H\right)t}\right]\;.$$

The time dependent transition amplitudes squared for the initial states to evolve to a state of the same or the opposite flavor are correspondingly given by

$$\begin{aligned} \left|\langle P^0|\mathcal{H}|\bar{P}^0\rangle\right|^2 \;&=\; \left|\frac{p}{q}\right|^2 |g_-(t)|^2\;, \\ \left|\langle \bar{P}^0|\mathcal{H}|P^0\rangle\right|^2 \;&=\; \left|\frac{q}{p}\right|^2 |g_-(t)|^2\;, \\ \left|\langle P^0|\mathcal{H}|P^0\rangle\right|^2 \;&=\; \left|\langle \bar{P}^0|\mathcal{H}|\bar{P}^0\rangle\right|^2 \;=\; |g_+(t)|^2\;, \end{aligned} \tag{2.18}$$

with

$$|g_\pm(t)|^2 \;=\; \frac{1}{2}e^{-\Gamma t}\left[\cosh\left(\frac{\Delta\Gamma}{2}\,t\right) \pm \cos\left(\Delta m\,t\right)\right]\;.$$

### 2.1.3 Neutral $B\bar{B}$ meson systems

Here we address specific characteristics of the two neutral $B$ meson systems – $B^0\bar{B}^0$ and $B_s\bar{B}_s$, which will be denoted $B^0_q\bar{B}^0_q$ ($q = d, s$), and point out appropriate formalism approximations. The symbol $B$ will also be used for denoting the bottom quantum number.

The off-diagonal mass matrix elements are responsible for the $B^0_q\bar{B}^0_q$ transitions. $M_{12}$ represents virtual transitions which provide the dominant contribution to the mixing amplitude. $\Gamma_{12}$ represents the real transitions through common decay modes. The latter alone implies that $|\Gamma_{12}| \ll \Gamma$. These common decay modes are furthermore Cabibbo (*i.e.* CKM) suppressed. If $\Gamma_{12}$ were to fully vanish, the relations (2.17) would yield $\Delta m = 2|M_{12}|$ and $\Delta\Gamma = 0$.

The following inequalities hold empirically for both systems

$$|\Gamma_{12}| \ll |M_{12}|, \qquad \Delta\Gamma \ll \Delta m \ , \tag{2.19}$$

such that an expansion in the respective ratios results in a good approximation. An approximate solution to (2.17) is accordingly provided by the following expansions

$$\begin{aligned}
\Delta m &= 2|M_{12}| \left[1 + \mathcal{O}\left(\left|\frac{\Gamma_{12}}{M_{12}}\right|^2\right)\right] \ , && (2.20)\\
\Delta\Gamma &= 2\,|\Gamma_{12}| \cos\phi_{12} \left[1 + \mathcal{O}\left(\left|\frac{\Gamma_{12}}{M_{12}}\right|^2\right)\right] \ , && (2.21)\\
\frac{q}{p} &= -e^{-i\phi_M}\left[1 - \frac{1}{2}\left|\frac{\Gamma_{12}}{M_{12}}\right| \sin\phi_{12}\right] + \mathcal{O}\left(\left|\frac{\Gamma_{12}}{M_{12}}\right|^2\right) \ , && (2.22)\\
&\text{with} \qquad \phi_M \equiv \arg(M_{12})\,, \quad \phi_{12} \equiv \arg\left(-\frac{M_{12}}{\Gamma_{12}}\right) \ .
\end{aligned}$$

For the $B^0$ system, $\Delta m_d \approx 0.75\Gamma_d$, while for the $B_s$ system existing experimental bounds give $\Delta m_s \gg \Gamma_s$. The existing experimental bounds on the fractional width differences place $\Delta\Gamma/\Gamma$ below 0.18 and 0.29 (95% C.L.) [1], respectively, for the $B^0$ and $B_s$ systems, while predicted theory bounds are correspondingly at less than 1% and less than 20% [10].

For both systems, $|\frac{q}{p}| = 1$ holds to a very good approximation. In effect, the difference

$$1 - \left|\frac{q}{p}\right|^2 \approx \mathrm{I}m\left(\frac{\Gamma_{12}}{M_{12}}\right) \tag{2.23}$$

is estimated to be $\sim \mathcal{O}(10^{-3})$ for the $B^0$ and $\lesssim\mathcal{O}(10^{-4})$ for the $B_s$ systems [1]. For these systems the $\Delta B = 2$ and the Cabibbo favored $\Delta B = 1$ effective operators are CP conserving.

The probability densities, denoted $\mathcal{P}$, for observing the initial (t=0) flavor eigentstates to decay at a later time $t$ with the opposite or the same flavor, following (2.18) in the limit of

$|\frac{q}{p}| = 1$ and negligible $\Delta\Gamma/\Gamma$, are given by

$$\mathcal{P}_{B_q^0 \to \bar{B}_q^0}(t) = \mathcal{P}_{\bar{B}_q^0 \to B_q^0}(t) = \frac{\Gamma}{2} e^{-\Gamma t} \left[1 - \cos(\Delta m \, t)\right] , \tag{2.24}$$

$$\mathcal{P}_{B_q^0 \to B_q^0}(t) = \mathcal{P}_{\bar{B}_q^0 \to \bar{B}_q^0}(t) = \frac{\Gamma}{2} e^{-\Gamma t} \left[1 + \cos(\Delta m \, t)\right] . \tag{2.25}$$

The frequency of flavor transitions corresponds, as explicitly shown, to the mass difference between the two mass eigenvalues of the system — $\Delta m$ constitutes therefore the target observable of time dependent flavor oscillation measurements.